On strong superadditivity for a class of quantum channels

Grigori G. Amosov*

Department of Higher Mathematics
Moscow Institute of Physics and Technology
Dolgoprudny 141700
RUSSIA

(Dated: February 1, 2008)

Given a quantum channel Φ in a Hilbert space H put $\hat{H}_{\Phi}(\rho)=\min\limits_{\rho_{av}=\rho}\Sigma_{j=1}^{k}\pi_j S(\Phi(\rho_j))$, where $\rho_{av}=\Sigma_{j=1}^{k}\pi_j\rho_j$, the minimum is taken over all probability distributions $\pi=\{\pi_j\}$ and states ρ_j in H, $S(\rho)=-Tr\rho\log\rho$ is the von Neumann entropy of a state ρ. The strong superadditivity conjecture states that $\hat{H}_{\Phi\otimes\Psi}(\rho)\geq\hat{H}_{\Phi}(Tr_K(\rho))+\hat{H}_{\Psi}(Tr_H(\rho))$ for two channels Φ and Ψ in Hilbert spaces H and K, respectively. We have proved the strong superadditivity conjecture for the quantum depolarizing channel in prime dimensions. The estimation of the quantity $\hat{H}_{\Phi\otimes\Psi}(\rho)$ for the special class of Weyl channels Φ of the form $\Phi=\Xi\circ\Phi_{dep}$, where Φ_{dep} is the quantum depolarizing channel and Ξ is the phase damping is given.

PACS numbers: 03.67.-a, 03.67.Hk

I. INTRODUCTION

A linear trace-preserving map Φ on the set of states (positive unit-trace operators) $\mathfrak{S}(H)$ in a Hilbert space H is said to be a quantum channel if Φ^* is completely positive ([7]). The channel Φ is called bistochastic if $\Phi(\frac{1}{d}I_H)=\frac{1}{d}I_H$. Here and in the following we denote by d and I_H the dimension of H, $dimH=d<+\infty$, and the identity operator in H, respectively.

Given a quantum channel Φ in a Hilbert space H put ([10])

$$\hat{H}_{\Phi}(\rho)=\min_{\rho_{av}=\rho}\sum_{j=1}^{k}\pi_j S(\Phi(\rho_j)),\qquad(1)$$

where $\rho_{av}=\sum\limits_{j=1}^{k}\pi_j\rho_j$ and the minimum is taken over all probability distributions $\pi=\{\pi_j\}$ and states $\rho_j\in\mathfrak{S}(H)$. Here and in the following $S(\rho)=-Tr(\rho\log\rho)$ is the von Neumann entropy of a state ρ. The strong superadditivity conjecture states that

$$\hat{H}_{\Phi\otimes\Psi}(\rho)\geq\hat{H}_{\Phi}(Tr_K(\rho))+\hat{H}_{\Psi}(Tr_H(\rho)),\qquad(2)$$

$\rho\in\mathfrak{S}(H\otimes K)$ for two channels Φ and Ψ in Hilbert spaces H and K, respectively.

The infimum of the output entropy of a quantum channel Φ is defined by the formula

$$\chi(\Phi)=\inf_{\rho\in\mathfrak{S}(H)}S(\Phi(\rho)).\qquad(3)$$

The additivity conjecture for the quantity $\chi(\Phi)$ states ([9])

$$\chi(\Phi\otimes\Psi)=\chi(\Phi)+\chi(\Psi)$$

for an arbitrary quantum channel Ψ. It was shown in ([10]) that if the strong superadditivity conjecture holds, then the additivity conjecture for the quantity χ holds too. Nevertheless the conjecture (2) is stronger than (3).

In the present paper we shall prove the strong superadditivity conjecture for the quantum depolarizing channel in prime dimensions of H. We also give some estimation from below for the quantity $\hat{H}_{\Phi\otimes\Psi}(\rho)$ for the certain class of Weyl channels Φ.

II. THE WEYL CHANNELS

Fix the basis $|f_j>\equiv|j>,\ 0\leq j\leq d-1$, of the Hilbert space H. We shall consider a special subclass of the bistochastic Weyl channels ([1, 2, 5, 6, 12]) defined by the formula ([2])

$$\Phi(\rho)=(1-(d-1)(r+dp))\rho+r\sum_{m=1}^{d-1}W_{m,0}\rho W_{m,0}^*\qquad(4)$$

$$+p\sum_{m=0}^{d-1}\sum_{n=1}^{d-1}W_{m,n}\rho W_{m,n}^*,$$

$\rho\in\mathfrak{S}(H)$, where $r,p\geq 0,\ (d-1)(r+dp)=1$ and the Weyl operators $W_{m,n}$ are determined as follows

$$W_{m,n}=\sum_{k=0}^{d-1}e^{\frac{2\pi i}{d}kn}|k+m\ mod\ d><k|,$$

$0\leq m,n\leq d-1$.

Consider the maximum commutative group $\mathcal{U}_d$ consisting of unitary operators

$$U=\sum_{j=0}^{d-1}e^{i\phi_j}|e_j><e_j|,$$

*Electronic address: gramos@mail.ru

where the orthonormal basis (e_j) is defined by the formula

$$|e_j>=\frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}e^{\frac{2\pi i}{d}jk}|k>,\ 0\le j\le d-1,$$

$\phi_j\in\mathbb{R},\ 0\le j\le d-1$. Notice that

$$<f_k|e_j>=\frac{1}{\sqrt{d}}e^{\frac{2\pi i}{d}jk},\ 0\le j,k\le d-1,$$

It implies that

$$|<f_k|e_j>|=\frac{1}{\sqrt{d}} \quad (5)$$

The bases (f_j) and (e_j) satisfying the property (5) are said to be mutually unbiased ([11]). It is straightforward to check that

$$W_{0,n}|e_j><e_j|W_{0,n}^*=|e_{j+n\ mod\ d}><e_{j+n\ mod\ d}|, \quad (6)$$

$0\le j,n\le d-1$.

It was shown in [2] that the Weyl channels (4) are covariant with respect to the group $\mathcal{U}_d$ such that

$$\Phi(UxU^*)=U\Phi(x)U^*,\ x\in\sigma(H),\ U\in\mathcal{U}_d.$$

Example 1. Put $r=p=\frac{q}{d^2},\ 0\le q\le 1$, then it can be shown ([1, 2, 5]) that (4) is the quantum depolarizing channel,

$$\Phi_{dep}(\rho)=(1-q)\rho+\frac{q}{d}I_H,\ \rho\in\mathfrak{S}(H), \quad (7)$$

$$\chi(\Phi_{dep})=-(1-\frac{d-1}{d}q)\log(1-\frac{d-1}{d}q)-(d-1)\frac{q}{d}\log\frac{q}{d}.$$

□

Example 2. Put $r=\frac{1}{d}(1-\frac{d-1}{d}q),\ p=\frac{q}{d^2},\ 0\le q\le\frac{d}{d-1}$, then (4) is q-c-channel ([9]). Indeed, under the conditions given above the channel $\Phi\equiv\Phi_{qc}$ can be represented as follows

$$\Phi_{qc}(\rho)=(1-\frac{d-1}{d}q)E(\rho)+\frac{q}{d}\sum_{n=1}^{d-1}W_{0,n}E(\rho)W_{0,n},$$

where

$$E(\rho)=\frac{1}{d}\sum_{m=0}^{d-1}W_{m,0}\rho W_{m,0}^*,$$

$\rho\in\mathfrak{S}(H)$ is a conditional expectation on the algebra generated by the projections $|e_j><e_j|,\ 0\le j\le d-1$. Taking into account (6) we get

$$\Phi_{qc}(\rho)=\sum_{j=0}^{d-1}Tr(|e_j><e_j|\rho)\rho_j,\ \rho\in\mathfrak{S}(H), \quad (8)$$

where

$$\rho_j=(1-\frac{d-1}{d}q)|e_j><e_j|+$$

$$\frac{q}{d}\sum_{k=1}^{d-1}|e_{j+k\ mod\ d}><e_{j+k\ mod\ d}|,$$

$0\le j\le d-1$,

$$\chi(\Phi_{qc})=-(1-\frac{d-1}{d}q)\log(1-\frac{d-1}{d}q)-(d-1)\frac{q}{d}\log\frac{q}{d}.$$

□

Proposition 1. *Suppose that the channel Φ has the form (4) and $p\le r\le\frac{1}{d}(1-d(d-1)p)$. Then, it can be represented as*

$$\Phi=\lambda\Phi_{dep}+(1-\lambda)\Phi_{qc},$$

$0\le\lambda\le 1$, where Φ_{dep} and Φ_{qc} are defined by the formulae (7) and (8), respectively.

Proof.

It follows from the condition $p\le r\le\frac{1}{d}(1-d(d-1)p)$ that there exists a number $\lambda,\ 0\le\lambda\le 1$, such that $r=\lambda p+(1-\lambda)\frac{1}{d}(1-d(d-1)p)$.

□

Suppose that the powers U^k of a unitary operator U in a Hilbert space H form a cyclic group of the order d. Fix the probability distribution $\pi=\{\pi_k,\ 0\le k\le d-1\}$, then the bistochastic quantum channel Ξ defined by the formula

$$\Xi(\rho)=\sum_{k=0}^{d-1}\pi_kU^k\rho U^{*k},\ \rho\in\mathfrak{S}(H),$$

is said to be *a phase damping.*

Proposition 2. *Suppose that the channel Φ has the form (4) and $p\le r\le\frac{1}{d}(1-d(d-1)p)$, then*

$$\Phi(\rho)=\Xi\circ\Phi_{dep}(\rho),\ \rho\in\mathfrak{S}(H), \quad (9)$$

where Φ_{dep} is the quantum depolarizing channel (7) and Ξ is the phase damping defined by the formula

$$\Xi(\rho)=\frac{1+(d-1)\lambda}{d}\rho+\frac{1-\lambda}{d}\sum_{m=1}^{d-1}W_{m,0}\rho W_{m,0}^*,\ \rho\in\mathfrak{S}(H),$$

$0\le\lambda\le 1$.

Remark. *The additivity conjecture for channels of the form (9) was proved in [1].*

Proof.

It is sufficiently to pick up the number λ defined in Proposition 1.

□

III. THE ESTIMATION OF THE OUTPUT ENTROPY

Our approach is based upon the estimate of the output entropy proved in [2]. Here we shall formulate the corresponding theorem without a proof for the convenience.

Theorem 2 ([2]). *Let $\Phi(\rho) = (1-p)\rho + \frac{p}{d}I_H,\ \rho \in \mathfrak{S}(H),\ 0 \le p \le \frac{d^2}{d^2-1}$, be the quantum depolarizing channel in the Hilbert space H of the prime dimension d. Then, there exist d orthonormal bases $\{e_j^s,\ 0 \le s,j \le d-1\}$ in H such that*

$$S((\Phi \otimes Id)(\rho)) \ge -(1-\frac{d-1}{d}p)\log(1-\frac{d-1}{d}p)- \quad (10)$$

$$\frac{d-1}{d}p\log\frac{p}{d} + \frac{1}{d^2}\sum_{j=0}^{d-1}\sum_{s=0}^{d-1}S(\rho_j^s),$$

where $\rho \in \mathfrak{S}(H\otimes K),\ \rho_j^s = dTr_H((|e_j^s><e_j^s|\otimes I_K)\rho) \in \mathfrak{S}(K),\ 0 \le j,s \le d-1$.

In the present paper our goal is to prove the following theorem.

Theorem. *Let Φ be the Weyl channel (4) in the Hilbert space of the prime dimension d satisfying the property $p \le r \le \frac{1}{d}(1-d(d-1)p)$. Then, for an arbitrary quantum channel Ψ in a Hilbert space K the inequality*

$$\hat{H}_{\Phi\otimes\Psi}(\rho) \ge -(1-\frac{d-1}{d}p)\log(1-\frac{d-1}{d}p)-$$

$$\frac{d-1}{d}p\log\frac{p}{d} + \hat{H}_{\Psi}(Tr_H(\rho)),\ \ \rho \in \mathfrak{S}(H\otimes K),$$

holds.

Remark. *Due to the covariance property of Φ_{dep} we get*

$$\hat{H}_{\Phi_{dep}}(\rho) = -(1-\frac{d-1}{d}p)\log(1-\frac{d-1}{d}p)-$$

$$\frac{d-1}{d}p\log\frac{p}{d} = const.$$

Hence, the theorem implies that

$$\hat{H}_{\Phi_{dep}\otimes\Psi}(\rho) \ge \hat{H}_{\Phi_{dep}}(Tr_K(\rho)) + \hat{H}_{\Psi}(Tr_H(\rho)),$$

$\rho \in \mathfrak{S}(H\otimes K)$.

Proof.

At first, let us prove the theorem only for the quantum depolarizing channel Φ_{dep}. Put $\tilde{\rho} = (Id\otimes\Psi)(\rho)$.

It follows from Theorem 2 of [2] that

$$S((\Phi_{dep}\otimes Id)(\tilde{\rho})) \ge -(1-\frac{d-1}{d}p)\log(1-\frac{d-1}{d}p)-$$

$$\frac{d-1}{d}p\log\frac{p}{d} + \frac{1}{d^2}\sum_{j=0}^{d-1}\sum_{s=0}^{d-1}S(\rho_j^s),$$

where $\rho \in \mathfrak{S}(H\otimes K),\ \rho_j^s = dTr_H((|e_j^s><e_j^s|\otimes I_K)\tilde{\rho}) \in \mathfrak{S}(K),\ 0 \le j,s \le d-1$.

Notice that

$$\frac{1}{d^2}\sum_{j=0}^{d-1}\sum_{s=0}^{d-1}S(\rho_j^s) = Tr_H(\tilde{\rho}) = \Psi(Tr_H(\rho)). \quad (11)$$

It follows from equality (11) that

$$\frac{1}{d^2}\sum_{j=0}^{d-1}\sum_{s=0}^{d-1}S(\rho_j^s) \ge \hat{H}_{\Psi}(Tr_H(\rho))$$

and we have proved the strong superadditivity conjecture for the quantum depolarizing channel.

The Weyl channel Φ satisfying the conditions of Theorem can be represented as a composition

$$\Phi = \Xi \circ \Phi_{dep}$$

in virtue of Proposition 2. It implies that

$$\hat{H}_{\Phi\otimes\Psi}(\rho) \ge \hat{H}_{\Phi_{dep}\otimes\Psi}(\rho)$$

due to the non-decreasing property of the von Neumann entropy. Thus, the result follows from the strong superadditivity property of the quantum depolarizing channel we have proved above.

□

IV. CONCLUSION

We have shown that our method introduced in [1, 2, 3] allows to prove the strong superadditivity conjecture for the quantum depolarizing channel. This method based upon the decreasing property of the relative entropy doesn't use the properties of l_p-norms of quantum channels. Thus, we suppose that the approach is fruitful for the future investigations in quantum information theory.

[1] Amosov G.G. Remark on the additivity conjecture for the depolarizing quantum channel. Probl. Inf. Transm. 42 (2006) 3-11. e-print quant-ph/0408004.

[2] Amosov G.G. On the Weyl channels being covariant with respect to the maximum commutative group of unitaries. e-print quant-ph/0605177 v.3.

[3] Amosov G.G. On the additivity conjecture for the Weyl channels being covariant with respect to the

maximum commutative group of unitaries. e-print quant-ph/0606040 v.3.

[4] Amosov G.G., Holevo A.S., Werner R.F. On some additivity problems in quantum information theory. Probl. Inf. Transm. 2000. V. 36. N 4. P. 24-34; e-print quant-ph/0003002.

[5] Datta N, Ruskai M.B. Maximal output purity and capacity for asymmetric unital qudit channels J. Physics A: Mathematical and General 38 (2005) 9785-9802. e-print quant-ph/0505048.

[6] Fukuda M., Holevo A.S. On Weyl-covariant channels. e-print quant-ph/0510148.

[7] Holevo A.S. On the mathematical theory of quantum communication channels. Probl. Inf. Transm. 8 (1972) 62 - 71.

[8] Holevo A.S. Some estimates for the amount of information transmittable by a quantum communications channel. (Russian) Probl. Inf. Transm. 9 (1973) 3 - 11.

[9] Holevo A.S. Quantum coding theorems. Russ. Math. Surveys 53 (1998) 1295-1331; e-print quant-ph/9808023.

[10] Holevo A.S., Shirokov M.E. On Shor's channel extension and constrained channels. Commun. Math. Phys. 249 (2004) 417-436.

[11] Ivanovich I.D. Geometrical description of quantum state determination. J. Physics A 14 (1981) 3241-3245.

[12] Karpov E., Daems D., Cerf N.J. Entanglement enhanced classical capacity of quantum communication channels with correlated noise in arbitrary dimensions. e-print quant-ph/0603286.